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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE DAMPING OF THE OSCILLATIONS OF SWINGING BODIES BY THE RESISTANCE OF THE AIR.

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By B. Osgood Peirce.

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When a body, free to turn about a fixed axis, like a horizontal pendulum, a suspended magnet, or the coil of a d'Arsonval galvanometer, is disturbed from a position of equilibrium, and is then allowed to swing under the action of a righting moment the intensity of which is proportional to the angular deviation of the body from the position of rest which it originally had, the damping effect of the resistance which the air offers to the motion is sooner or later made evident by a reduction in the amplitude of the swings. In many cases the phenomena can be quantitatively explained, with an approximation quite good enough for every practical purpose, if one assumes that the resisting couple has a moment equal at every instant to the product of a constant of the apparatus and the angular velocity which the body then has; and more than seventy years ago Gauss and W. Weber gave an exhaustive mathematical treatment,[1] based upon this hypothesis, of the behavior of such swinging magnets as they employed in their magnetic measurements at Göttingen. It appeared from their analysis, which in simplified form is given in most modern treatises on Physics, that if the resistance follows the law stated above, the ratio of any two successive elongations of the magnet must have a constant value; and they used the natural logarithm (λ) of this ratio, under the name of the "logarithmic decrement" of the motion, in many of their equations.

The resistance which air, under given conditions of temperature, pressure, and confinement, offers to a body of given form and dimensions, moving through it at a *uniform* velocity, v, has been studied by

1 Gauss, Resultate des Magnetischen Vereins, 1837. W. Weber, Resultate des Magnetischen Vereins, 1837, 1838; Maassbestimmungen, **2**; Math.-phys. Abhandlungen der K. Sächs. Gesellschaft, 1852. Du Bois-Reymond, Monatsberichte der Berl. Akad., 1869, 1870.

a great number of experimenters under a great variety of physical conditions, and a resumé of the results at which they have arrived can be found in the articles of Finsterwalder on Aërodynamik and of Cranz on Ballistik in the fourth volume of the Encyklopädie der Mathematischen Wissenschaften.[2]

That under otherwise given conditions the air resistance, when v is large, is a complicated function of v, is shown by the practical formulas based on experiments made with rotating projectiles of the standard Krupp form. For a projectile of this kind of given size, in free air, the expressions are av^2, bv^3, cv^5, dv^3, ev^2, $fv^{1.7}$, $gv^{1.55}$, according as v, measured in meters per second, lies in one or other of the intervals between the values 50, 240, 295, 375, 419, 550, 800, and 1000. The constants are different for projectiles of different diameters and vary with the temperature of the air, the barometric pressure, and other circumstances.

In order to determine the resistance which the air offers to a given body moving uniformly through it at a comparatively small velocity, v, many different observers have made use of the whirling table in some form. The phenomenon to be studied is in any case a very complex one, since the moving body drags with it, as it moves, a certain mass of air, and the viscosity of the air contributes an uncertain amount to the quantity to be measured. It appears, however, from the experiments of Schellbach, von Loessl, Langley, Recknagel, Hagen, and others,[3] that when proper corrections have been made for the effect of the wind which the table takes with it as it turns, the air resistance varies as the square of the velocity [4] for all values of v between 50 and 0.2. For velocities much less than 20 centimeters per second the viscosity of the air appears to determine the resistance which is approximately proportional to the velocity. It is well to remember that a solid sphere, to take a concrete example, moving in an infinite homogeneous liquid at rest at infinity, in a straight line, with *constant*

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[2] Leipzig, B. G. Teubner, 1903.

[3] Schellbach, Ann. d. Phys., **143**, 1871. Recknagel, Zeitschrift d. Ver. deutsch. Ing., **30**, 1886. F. v. Loessl, Die Luftwiderstandsgesetze. Langley, Experiments in Aërodynamics. Cranz, Aeussere Ballistik, Leipzig, 1895. Thiesen, Ann. d. Phys., **26**, 1885. Mach und Salcher, Wiener Berichte, 1887, 1889.

[4] Mohn, Grundzüge der Meteorologie, Zweite Auflage, p. 137: "Durch vergleichende Versuche über Druck und Geschwindigkeit des Windes, hat man gefunden dass der Winddruck dem Quadrate der Geschwindigkeit proportional ist." On page 138, however, the pressure of the wind in kilograms per square meter is given as 0.15, 1.87, 5.96, 15.27, 34.35, 95.4, according as the velocity in meters per second is 0.5, 4, 7, 11, 17, or 28.

velocity, would encounter no resistance from the liquid if there were no viscosity; but that even in a homogeneous, perfect liquid, a sphere moving with *changing* velocity would meet with a resistance from the liquid, and the inertia of the sphere would in consequence of this be apparently increased in a manner which could be mathematically accounted for in the equation of motion of the sphere, if the mass of the sphere were increased by half the mass of the displaced liquid.

If at the point (x, y, z) in a viscous fluid at the time t the components of the velocity are u, v, w, if the applied body forces which urge the fluid have the components X, Y, Z, if ρ is the density, and if μ represents a constant of the fluid which measures its coefficient of viscosity, the equations of motion of the fluid as established by Navier and Poisson [5] are usually written in the forms:

$$\rho\left(\frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial x} + v\cdot\frac{\partial u}{\partial y} + w\cdot\frac{\partial u}{\partial z}\right) = \rho X - \frac{\partial p}{\partial x} + \tfrac{1}{3}\mu\cdot\frac{\partial \varpi}{\partial x} + \mu\cdot\nabla^2(u),$$

$$\rho\left(\frac{\partial v}{\partial t} + u\cdot\frac{\partial v}{\partial x} + v\cdot\frac{\partial v}{\partial y} + w\cdot\frac{\partial v}{\partial z}\right) = \rho Y - \frac{\partial p}{\partial y} + \tfrac{1}{3}\mu\cdot\frac{\partial \varpi}{\partial y} + \mu\cdot\nabla^2(v),$$

$$\rho\left(\frac{\partial w}{\partial t} + u\cdot\frac{\partial w}{\partial x} + v\cdot\frac{\partial w}{\partial y} + w\cdot\frac{\partial w}{\partial z}\right) = \rho Z - \frac{\partial p}{\partial z} + \tfrac{1}{3}\mu\cdot\frac{\partial \varpi}{\partial z} + \mu\cdot\nabla^2(w), \tag{1}$$

where $$\varpi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

and p represents the arithmetical mean of the normal pressures on any three mutually perpendicular planes through the point (x, y, z).

Using these equations, Stokes, in a paper [6] presented to the Cambridge Philosophical Society in December, 1850, determined the resistance which a sphere making small harmonic oscillations of complete period T, in an infinite viscous liquid, would encounter, and showed that if θ represented the distance of the centre of the sphere from its mean position at the time t, the value of this resistance would be

$$\left(\frac{1}{2} + \frac{9}{4af}\right) M'\cdot\frac{d^2\theta}{dt^2} + \frac{9\pi M'}{2afT}\left(1 + \frac{1}{af}\right)\cdot\frac{d\theta}{dt}, \tag{2}$$

or

$$M''\frac{d^2\theta}{dt^2} + 2m\frac{d\theta}{dt}, \tag{3}$$

[5] Navier, Mémoire de l'Académie des Sciences, **6**, 1822. Poisson, Journal de l'École Polytechnique, **13**, 1829.

[6] Stokes, Mathematical and Physical Papers, II.

where a is the radius of the sphere, M', the mass of the displaced liquid, and $f^2 = \pi\rho/\mu T$: M_0 is the mass of the sphere.

Such a sphere, oscillating under the action of this resistance and a restoring force $(b^2\theta)$ proportional to the displacement, would have an equation of motion of the form

$$M, \frac{d^2\theta}{dt^2} + 2m \cdot \frac{d\theta}{dt} + b^2\theta = 0, \tag{4}$$

where $M = M_0 + M''$: all the coefficients are to be considered contant, since b^2 is fixed, but they would be different for a different period of oscillation.

For an infinitely long cylinder of revolution also, oscillating in a viscous liquid, in a direction perpendicular to the axis of the cylinder, Stokes found an equation of motion of this same familiar form which had long been used to explain the behavior of pendulums, though it had been founded on a theory quite different from his. As early as 1828 Bessel [7] had pointed out the necessity of allowing for the inertia of the air which accompanies a pendulum in its motion, and the work of Sabine, Dubuat, Poisson, Baily, Plana, South, and others, had made it clear that in practical cases the moment of inertia of the swinging system might be twice that of the pendulum bob, and that the "resistance" of the air might be accounted for in many practical cases by assuming it to be proportional to the first power of the angular velocity. This equation had been used by Gauss for determining the motion of swinging bar magnets, as has been already mentioned, and it still forms the foundation of much modern work, as, for instance, that on the properties of damped d'Arsonval galvanometers.[8]

If, however, a swinging magnet presents to the air a relatively large surface, or if the magnet is provided with a large mica damping vane, it often happens that the resistance of the air cannot be satisfactorily explained on the assumption that it is proportional to the angular velocity at every instant, and that at the beginning of the motion it

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7 Bessel, Untersuchungen über die Länge des einfachen Secunden Pendels, Berlin, 1828. Bottomley, Phil. Mag., **23**, 1887. Graetz, Reibung, Winkelmann's Handbuch der Physik, I. O. E. Meyer, Pogg. Ann., **113**, 1861; **125**, 1863; **142**, **143**, 1871; **148**, 1873. Wied. Ann., **23**, 1887. Kundt und Warburg, Pogg. Ann., **155**, 1875. Crookes and Stokes, Proceedings Royal Society, 1888.

8 Dorn, Ann. der Physik, **17**, 1882; **35**, 1888. F. Kohlrausch: Ueber die Inconstanz der Dämpfungsfunction eines Galvanometers und ihren Einfluss auf die Absolute Widerstandsbestimmung mit dem Erdinductor, Ann. der Phys., **26**, 1885. Schering, Ann. der Phys., **9**, 1880. Jaeger, Instrumentenkunde, 1903. Dorn, Ann. der Physik, **17**, 1882.

seems to be much more nearly proportional to the square of the angular velocity. It will be convenient, therefore, to consider first the manner in which the amplitude of an oscillating body would decrease if the motion were resisted by a couple of moment proportional to the square of the angular velocity. A roughly approximate solution of this problem was printed by Poisson in 1811, but is not accurate enough for practical purposes. We shall do well to attack it in another way.

If θ is the angular deviation in radians of the moving body from the position of equilibrium, and $b^2\theta$ the restoring moment, the moment of the couple due to the resistance of the air is of the form $2a(d\theta/dt)^2$; and if K represents the moment of inertia of the swinging system, the equation of motion is

$$K \cdot \frac{d^2\theta}{dt^2} + 2a\left(\frac{d\theta}{dt}\right)^2 + b^2\theta = 0, \tag{5}$$

or

$$\frac{d^2\theta}{dt^2} + 2a\left(\frac{d\theta}{dt}\right)^2 + \beta^2\theta = 0, \tag{6}$$

when the body is swinging in the positive direction.

If for $d\theta/dt$ we write ω, $d^2\theta/dt^2$ is equal to $\omega \cdot d\omega/d\theta$, and the equation becomes

$$\omega \cdot d\omega + (2a\omega^2 + \beta^2\theta)d\theta = 0, \tag{7}$$

which will become exact if we multiply through by $e^{4a\theta}$, so that,

$$\omega^2 = 2c \cdot e^{-4a\theta} + \frac{\beta^2}{8a^2} - \frac{\beta^2\theta}{2a}, \tag{8}$$

or

$$\omega^2 = 2c\,e^{-k\theta} + m - mk\theta, \tag{9}$$

where c is a constant of integration.

If $-\theta_0$ is the value of the angular deviation at any elongation on the negative side, and if θ_1 is the next elongation on the positive side, then, for the same value of c,

$$2c\,e^{k\theta_0} + m + mk\theta_0 = 0, \tag{10}$$

$$2c\,e^{-k\theta_1} + m - mk\theta_1 = 0, \tag{11}$$

or

$$(1 + k\theta_0)\,e^{-k\theta_0} = (1 - k\theta_1)e^{+k\theta_1}, \tag{12}$$

where $k = 4a$. This equation does not involve β.

For swings of large amplitude, it is easy to find θ_1 graphically, when k and θ_0 are given, by aid of this last equation. When θ_0 is small, however, we may, in any practical case, develop each number of (12) in a very convergent power series of which we need keep only terms of order lower than the fourth.

This procedure gives the equation

$$2k(\theta^3_0 + \theta^3_1) - 3(\theta^2_0 - \theta^2_1) = 0, \tag{13}$$

which is satisfied when $\theta = -\theta_0$ and from this we may find, by aid of a second development, the very approximate result,

$$\theta_1 = \theta_0 - \tfrac{2}{3} k\theta_0^2. \tag{14}$$

If terms of the fourth order are kept, we may obtain the expressions

$$\theta_1 = \theta_0 - \tfrac{2}{3} k\theta^2_0 + \tfrac{4}{3} k^2\theta^3_0, \tag{15}$$

$$\theta_2 = \theta_0 - \tfrac{4}{3} k\theta^2_0 + \tfrac{16}{9} k^2\theta^3_0,$$

but for most practical purposes (14) is quite accurate enough.

After the swinging system has come momentarily to rest at the elongation $-\theta_0$, it moves in the positive direction with an angular velocity which increases to a maximum at a position determined by the constants of the motion, and has the value ω_0 when $\theta = 0$.

It is easy to see from (3) that

$$\omega_0^2 = 2c + m, \tag{16}$$

and from (9) that

$$2c = -m(1 + k\theta_0)\, e^{-k\theta_0}, \tag{17}$$

so that

$$\omega_0^2 = m - m(1 + k\theta_0)e^{-k\theta_0}; \tag{18}$$

and it is evident that ω_0 is greater, other things being given, the greater the amplitude of the motion; that is, the greater the value of θ_0. Equation (16) shows, however, that the greatest value which ω_0 can have is $\sqrt{m}$, and it is interesting to determine what elongation on the positive side of the zero point corresponds to this angular velocity at $\theta = 0$.

If in (12) we suppose θ_0 to grow large without limit, θ_1 approaches the limit $1/k$, and it appears that however great the angle through which the swinging system may have been turned out of the position of equilibrium at the outset, the amplitude of the next elongation cannot be greater than $1/k$th of a radian, and the next turning point to this (on the same side of the zero as the original disturbance) must come at an angular distance from the position of equilibrium not greater than about $0.594/k$ radians. The subsequent swings decrease regularly in amplitude in such a manner as to make the logarithmic decrement decrease towards zero. At any time during the motion the determination of two successive amplitudes serves to determine k through (12), for it is easy to solve the transcendental equation to any desired accuracy.

If we differentiate (6) with respect to t, and represent $d\omega/dt$ by r, we shall get the equation

$$\frac{rdr}{4\,ar + \beta^2} + \omega d\omega = 0, \tag{19}$$

or

$$r - \frac{\beta^2}{4\,a} log\left(r + \frac{\beta^2}{4\,a}\right) = C - 2\,a\omega^2, \tag{20}$$

and C, the constant of integration, may be determined from a consideration of the fact that when $\omega = 0$, θ is $-\theta_0$.

If a swinging system oscillates about a position of equilibrium under the action of a righting moment proportional to the deviation and a resisting couple proportional during the whole motion to the *first* power of the instantaneous angular velocity, the equation of motion has the familiar form

$$\frac{d^2\theta}{dt^2} + 2\,a\frac{d\theta}{dt} + \beta^2\theta = 0. \tag{21}$$

If $\rho^2 = \beta^2 - a^2$, and if m and n are the roots of the equation

$$x^2 + 2\,ax + \beta^2 = 0, \tag{22}$$

$$m = -\,a + \rho i, \qquad n = -\,a - \rho i,$$

and we have $$\theta = e^{-at}(L\cos\rho t + M\sin\rho t), \tag{23}$$

or $$\theta = Ae^{-at}\sin(\rho t - e), \tag{24}$$

where A and e are constants of integration. If, using t and θ as coordinates, we plot (24), it is clear that the curve $\theta = Ae^{-at}$ touches the curve $\theta = Ae^{-at}\sin(\rho t - e)$ when $\rho t - e = (2\,k + \frac{1}{2})\pi$, so that if the time be counted from the date of one of these points of tangency, the corresponding solution of (21) may be written in the form

$$\theta = Be^{-at}\cos\rho t. \tag{25}$$

The complete period of the oscillation (T) is $2\pi/\rho$. The ratio of the amplitudes at two consecutive elongations is $e^{a\pi/\rho}$ and the logarithmic decrement is $a\pi/\rho$. The ratio of the amplitudes at two consecutive elongations on the same side of the position of equilibrium is $e^{2a\pi/\rho}$, and we have

$$a = 2\lambda/T, \qquad \beta^2 = 4(\pi^2 + \lambda^2)/T^2. \tag{26}$$

The maxima of the curve (24) occur at times defined by the equation

$$\tan(\rho t - e) = \rho/a; \quad \text{or} \quad \sin(\rho t - e) = \rho/\beta,$$

and the curve $$\theta = \frac{A\rho}{\beta} e^{-at}$$

passes through all these points, which are spaced at equal time intervals T.

If, then, the curve which represents as a function of the time (t) the deviation (θ) of a swinging body from the position of equilibrium be drawn, and if the motion be of the kind defined by the equation (21), the maxima will be spaced at equal time intervals, and it will be possible to pass through all the crests a curve of the family $\theta = Ce^{-at}$ where C and a are constants. It is easy to see whether or not this last condition is satisfied in any given case, if one has measured a series of successive amplitudes on the same side (d_1, d_2, d_3, d_4, d_5, . . . d_r). If we measure t from the date of the first of these elongations, the desired curve must have an equation of the form $\theta = d_1 e^{-at}$, and aT may be determined from any other amplitude (say the kth) for

$$d_k = d_1 e^{-(k-1)aT}.$$

If the value of aT thus found be the same for all values of k, the condition is satisfied. Sometimes when the period of the oscillation is extremely short, the maximum points seem to form a continuous curve, unless the diagram be much drawn out horizontally. In such a case as this one may use, in making the test just described, not a series of successive amplitudes on the same side of the position of equilibrium, but points on the curve, taken at convenient values of t equally spaced.

The Damping of the Quick Oscillations of a Light System suspended between two Stretched Wires by the Resistance of the Air and Frictional Forces in the Wire.

It will appear from the observations recorded in this paper that if a small magnetic needle be mounted horizontally with a minute galvanometer mirror upon a short, stiff, vertical piece of wire or glass filament stretched between two vertical pieces of fine wire, and if the needle be turned horizontally out of its position of rest through an angle of say 5° and then allowed to oscillate, the curve drawn through the crests of the oscillations as represented on a photograph record will usually not coincide exactly with any exponential curve of the family mentioned above. If a curve of this family be drawn nearly through a

number of crests in the middle of the diagram, it will usually fall somewhat below the observed curve at each end. It will be convenient to instance a few typical cases at the outset.

I. Figure 1 (Plate 1) is a copy of a photographic record obtained from a short-period mirror galvanometer. The one-centimeter-long needle of this instrument, made of watch spring, was mounted on a short, stout, inflexible piece of glass fibre, together with a minute bit of very thin mirror, and the fibre was suspended, like the coil of a marine d'Arsonval galvanometer,[9] between two pieces of extremely fine gimp, under gentle tension. The light from an electric projecting lantern about twenty feet from the galvanometer, shining through a small hole in a brass plate used as a lantern slide, fell upon the galvanometer mirror, and a sharp image of the hole was formed on a piece of sensitive paper on a horizontal revolving drum at a considerable distance from the mirror. The needle was first deflected a little off scale by a steady current sent through the coils of the galvanometer when the light was screened, the screen was then removed and a record of the manner of decay of the amplitude of the excursions of the needle obtained when the galvanometer circuit had been suddenly broken. The moment of the couple due to the mutual action of the magnet and the earth's field was relatively inappreciable. Three different drums and three pieces of chronograph clock work were used in making the records discussed in this paper, but for the fastest speeds an electric motor driving a worm gear accurately cut for the purpose by Mr. G. W. Thompson, the mechanician of the Jefferson Laboratory, was employed, and this left nothing to be desired. The apparatus was put together by Mr. John Coulson, who helped me in all the work and took many of the photographic records. Most of these records, of which I have a very large number, were about 50 cms. long and 20 cms. wide, but much larger ones could be obtained if desirable.

On one of the photographs taken with the apparatus just described a series of measurements of the amplitudes of the oscillations, as depicted on the diagram, were made at times, represented by whole centimeters from the time origin, on the figure. The successive values for the excursions were: 1260, 1006, 791, 646, 521, 420, 349, 280, 231, 190, 159, 131, on a scale of equal parts, and, at the scale distance used, these numbers were accurately proportional to the angular amplitudes of the needle at the times concerned. If, then, the resistance to the oscillations were proportional to the angular velocity, it should be possible to draw a curve of the family $y = A \cdot e^{-at}$ the successive ordinates of

[9] See *M*, Figure 3.

which, taken at the proper time interval (T), should have the lengths indicated above.

If, however, we assume that when $t = 0$, $y = 1260$, and use the other numbers given above, in succession for determining aT, we get for this product the different values 0.225, 0.238, 0.224, 0.221, 0.220, 0.217, 0.214, 0.212, 0.210, 0.207, and it is evidently impossible to find the curve sought exactly. The differences, while much too great to be accidental, are intrinsically not very large, and a curve of the family $y = A \cdot e^{-at}$ may be drawn which will pass through the fourth, fifth, and sixth points, and the ordinates of which at the ends of the series will be 1231 and 116, instead of 1260 and 131. Corresponding ordinates of the observed and calculated curves are shown in the following table. The calculated curve has, as a whole, a less curvature than the observed curve, and the ratio of any excursion to the next decreases slightly with the time. The period was about 1/100th of a second.

TABLE I.

Observed.	Calculated.	Observed.	Calculated.
1260	1231	349	339
1006	993	280	273
791	801	231	221
646	646	190	178
521	521	159	143
420	420	131	116

In the case of this particular quickly oscillating system, therefore, the first double amplitude of which was not larger than 6°, the motion can be explained during a considerable part of its course with fair approximation on the assumption that the resistance due to the air and to frictional forces in the fibre is proportional to the angular velocity. The deviations from this law, while real, are not greater than one often finds in the motion of a suspended magnet or the coil of a d'Arsonval galvanometer, when swinging slowly over a small arc. Indeed two d'Arsonval galvanometers of the same type and apparently very like each other may depart from the usual law in opposite directions if the periods are long; in one the logarithmic decrement may grow larger as the amplitudes of a long series of swings decay, while in the other it may become smaller. In the case of a galvanometer of this kind in the Jefferson Laboratory, the ratio of one excursion to the next *increased* from 1.063 to 1.086 in an hour and a quarter, while the amplitude decreased to about four tenths of its original value. The complete period of swing was 158 seconds. A similar galvanometer in the

same room has a coil the swings of which decay at a *decreasing* rate as the amplitudes grow less.

In his Anleitung zur Bestimmung der Schwingungsdauer einer Magnetnadel (1837), Gauss describes a suspended magnet the logarithmic decrement of the swings of which increased on a certain occasion from 1168×10^{-6} to 1301×10^{-6} in 422 oscillations. The actual value of the logarithmic decrement for this magnet and for a given amplitude varied from day to day, being usually smaller in cloudy weather.

II. After a number of records had been made like that reproduced in Figure 1, a small vertical mica damping vane of about 3 square centimeters area, was fastened symmetrically to the little glass rod which carried the mirror of the swinging system, and a new series of records were obtained. The restoring moment was the same as before, but the moment of inertia had been increased somewhat, as well as the resistance due to the air. Under these circumstances the period was much longer than before, while the manner of decay of the amplitudes was much the same. Figure 2 (Plate 1) represents on a reduced scale one of the smaller photographs. Figure A was plotted from a large record in which the crests of successive oscillations were 4.5 millimeters apart at the beginning of the diagram and nearly 4.9 millimeters at the end. Such a gradual change of period during the motion often accompanies the swinging of a magnet under the torsional forces of a stretched wire.[10]

The values, in ten thousandths of a radian, of a number of successive amplitudes, as obtained from the photograph, were: 597, 556, 518, 481, 448, 419, 390, 367, 341, 320, 300, 280, 262, 246, 230, 217, 203, 190, 179, 168, 159, 148, 140, 132, 124, 117, 111, 104, 98, 92.

These numbers, used as ordinates of points with equally spaced abscissas, give a curve of the form shown in Figure A by the full line WHCDK. The dotted line VCDG shows a curve of the family $y = A \cdot e^{-at}$, which coincides almost exactly with the full line between the points C and D.

The curves HT, CL represent attempts to determine the constants of an equation of the form (6) which should yield a curve of amplitudes like the observed curve. Both HT and CL pass exactly through two adjacent points of the line WHCDK, and the other points were determined by a series of applications of the equation (8). Some of the characteristics of certain of the records which I obtained resemble those of oscillations under a resistance proportional to the square of

10 Guthe, Physical Review, 1908.

the angular velocity, but it is evident that in the case here considered the resistance does not nearly follow this law. We may notice that according to Poisson's rather rough approximation, HT and CL would be straight lines.

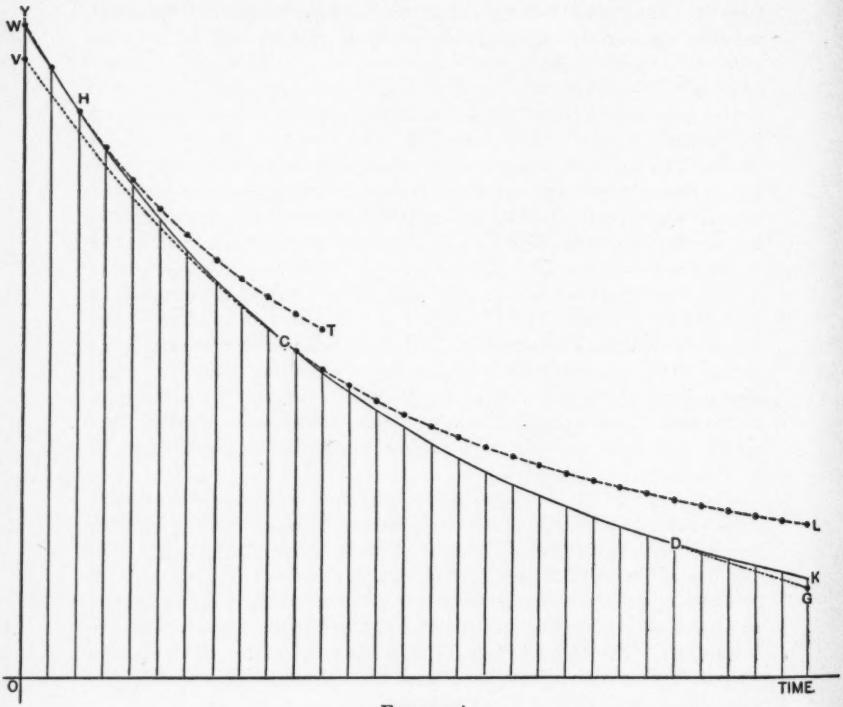

FIGURE A.

III. A new suspended system was then made of two 15 millimeter long magnetic needles mounted horizontally, one over the other (together with a small mirror), on a short bit of glass fibre stretched between two short lengths of No. 36 steel wire. The restoring forces came from the torsional forces in these wires. The mirror and needles together exposed to the air a resisting surface of less than a square centimeter area. The period was about 1/48th of a second. The

numbers in the first column of the next table show the lengths of successive ordinates (taken at equal time intervals) of the curve drawn on the photographic record through the crests of the oscillations. The next column gives the lengths of the corresponding ordinates of a curve of the family $A \cdot e^{-at}$ drawn exactly through the fifth and tenth crests. The very tips of the needles at the beginning of the motion passed over about 10 centimeters of path per second.

TABLE II.

1705	1509	418	411
1445	1341	379	365
1230	1141	342	324
1075	1058	313	288
940	940	282	256
828	835	255	227
734	742	230	202
652	658	208	179
582	586	185	159
520	520	174	142
465	462		

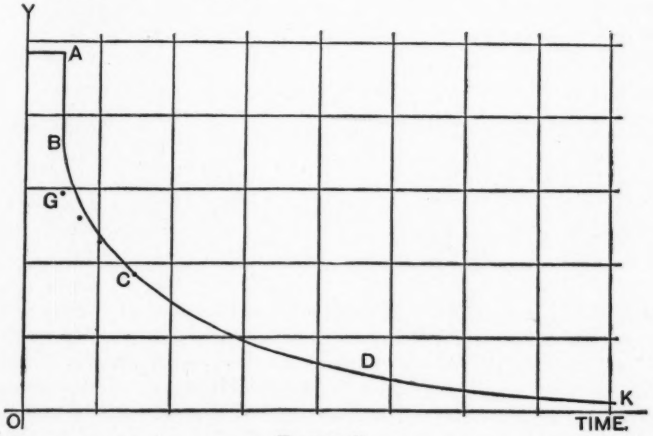

FIGURE B.

IV. Figure B shows the manner of decay of the oscillations of a light suspended system under the action of very strong restoring forces. A small mirror and two 15 millimeter long watch-spring magnets were

mounted on a square vertical mica vane, of about 3 square centimeters area, which was fastened symmetrically on a slender but stiff bit of glass filament. The filament was stretched between two pieces of No. 36 B. & S. steel wire about 2 centimeters long. The righting moment was due partly to the torsional forces in the wire and partly to a strong electromagnetic field about the needles. When the circuit of the magnetic field used to deflect the needle through the initial angle θ_0 was suddenly broken, the vane and its belongings moved quickly (in perhaps 1/250th of a second) through the position of equilibrium and out on the other side to a turning point corresponding to a deviation of about three fourths of θ_0. After this the amplitudes decreased slowly and continuously. The curve drawn through the crests of the oscillations consists at the start of a vertical line, as it would if, for instance, the resistance followed the law of the square of the angular velocity. After a short time, however, the curve, like most of those which I have obtained, follows more nearly a course which corresponds to the equation $y = A \cdot e^{-at}$. The numbers in the next table show well enough what the character of the agreement is. The first column gives ordinates of the photographic record taken at equal time intervals. The second column gives corresponding ordinates of a curve of the family $y = A \cdot e^{-at}$ which falls in very nearly with the first curve for a portion of the middle of its course.

TABLE III.

4840–3750	2950	1100	1095
2950	2695	1012	1001
2560	2463	930	915
2295	2251	862	836
2065	2057	800	764
1880	1880	745	698
1710	1718	695	639
1560	1570	645	583
1430	1435	600	533
1315	1312	...	...
1200	1200	390	310

V. Figure C represents curves taken with this apparatus when the filament was made of a piece of manganine wire. One curve is here displaced an arbitrary amount with respect to the other, for purposes of comparison. The sudden drop (ST) from the original deflected position to one of much smaller displacement, after which the de-

crease of amplitude is gradual, is clearly shown. The two curves show different values of the original deflection.

The Damping of the Slow Oscillations of a d'Arsonval Galvanometer Coil, which is wound on a Nonmetallic Core, and is swinging between the Poles of its Magnet.

If the coil of a d'Arsonval galvanometer be wound on a wooden spool, and if its circuit be open, the damping of its oscillations is due principally, unless the copper wire is magnetic, to air resistance, and

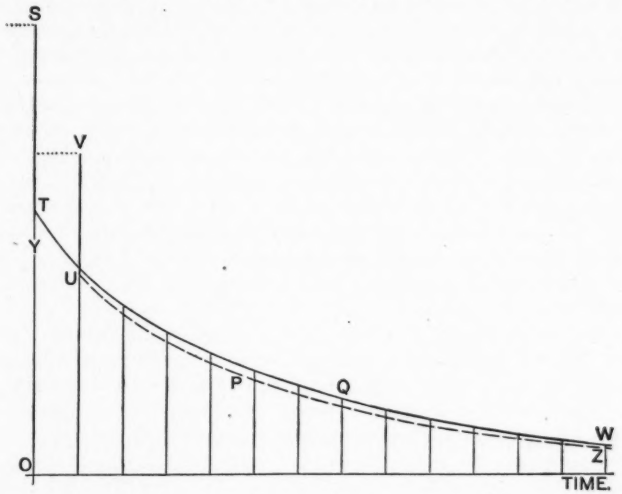

Figure *C*.

only slightly to frictional forces within or at the surface of the gimp from which the coil hangs. When, however, the circuit of the coil is closed through an outside resistance x, electromagnetic damping is added, and the damping coefficient of the motion is larger than before, or, if x is small enough, the motion ceases to be periodic. In many instances it is possible and desirable to damp the coil critically, but this is sometimes impracticable, — as, for instance, in such instruments of long period (400 or 500 seconds) as are used in testing massive iron cores, — and there are certain kinds of absolute measurements where a relatively undamped instrument is preferable. The throw of

a d'Arsonval galvanometer due to a given change of the flux of magnetic induction through its circuit is usually to be quantitatively explained only by attributing to the resistance of the circuit a value much greater than the real one. This apparent resistance [11] may be many times as great as the real resistance; its value depends upon the constants of the motion of the coil, and it not infrequently happens that a knowledge of these "constants" is important, even though the amplitudes do not always decrease exactly according to the assumption that the resistance to the motion is equivalent to a couple of moment proportional to the angular velocity.

If a coil of the ordinary Ayrton-Mather form, without a damping vane, swing between the poles of its magnet with the coil circuit open, the amplitude generally decreases slowly, and if the coil be hung successively by pieces of gimp of different lengths or stiffnesses, the period changes with the restoring moment, and the damping coefficient (a) remains small, though it often changes somewhat with the amplitude. If with a given suspension we determine the quantity a in the equation $y = A \cdot e^{-at}$ from two amplitudes of about 5° near the beginning of the motion, and then from two amplitudes of about 2° after the coil has made twenty or thirty swings, the latter value will usually be sensibly smaller than the other, but the difference is not very great unless the restoring force is weak, as it is in very sensitive instruments.

VI. In the case of a certain galvanometer of the Ayrton-Mather type which I studied at length, the value of a fell from 0.00403 to 0.00356 as the motion progressed, when a piece of very fine steel gimp was used to hang the coil. When stiff gimp was employed, the value of a remained much more nearly constant while the amplitude decreased, and was nearly the same for different lengths of the gimp. The first column in the next table shows the period as determined principally by the stiffness of the gimp, the second column gives the corresponding value of a determined after twenty or thirty swings had been executed and the double amplitude had fallen below 4°.

TABLE IV.

T.	Damping Coefficient.
2.59	0.0029
3.62	0.0028
4.57	0.0031

[11] Robinson, The Electrician, 1901. White, Physical Review, 1904. Peirce, These Proceedings, 1906.

The resistance of the instrument was about 21 ohms, but a considerable fraction of this was in the gimp.

When the coil circuit was closed by a resistance of 400 ohms, and the coil was hung successively by several different pieces of gimp of different lengths, the damping coefficient (a) slowly decreased as the amplitude decreased, so that the logarithmic decrement was not quite constant during the whole motion in any case, but the value of a for a double amplitude of say 4° was practically the same for widely different periods.

The next two tables show the results of measurements of a good number of photographic records. In the first case, as has been said, the outside resistance of the circuit was 400 ohms, in the second case it was 200 ohms.

TABLE V.

T.	Damping Coefficient.
9.28	0.0113
4.57	0.0113
3.62	0.0113
2.61	0.0114

TABLE VI.

T.	Damping Coefficient.
7.58	0.0193
4.57	0.0192
3.62	0.0192
2.61	0.0191

If the coil was deflected out of its position of equilibrium through an angle of perhaps 10°, and was then suddenly released, the amplitude fell at once to a much smaller value, especially when the coil was closed through a resistance of say 400, and then decreased gradually in much the same manner as the swings represented by Figure 5. The phenomenon is, however, not so marked as when the damping is fairly large and due wholly to air resistance.

When the circuit of the coil of a d'Arsonval galvanometer of the form described is closed through an outside resistance, x, so that the whole resistance is $(g + x)$, the damping coefficient of the motion is theoretically the sum of the corresponding coefficient when the circuit is open and the coefficient which the electromagnetic damping would cause if the air damping were absent, and this last should be proportional reciprocally to the apparent resistance $(g' + x)$ of the circuit, where g' is usually considerably larger than g. A set of five photo-

graphic records were obtained with the coil mentioned above when it was suspended by a certain short piece of wire which gave the system a period of about 2.60 seconds. The next table shows (1) the values of x, (2) the corresponding values of a determined by a series of measurements of the diagrams from amplitudes not greater than 4°, (3) the values which the damping coefficient (a') would have if the air damping were absent, as calculated by aid of Table IV, and finally the reciprocal (y) of a'. Since a' should theoretically be of the form

$$\frac{.1}{y} = \frac{b}{x + g'}, \tag{29}$$

if the observed values of x and y be plotted, the locus should be a straight line the intercept of which on the axis of abscissas is the value of g'.

TABLE VII.

x.	a.	a'.	y.
400	0.0114	0.0085	11.76
200	0.0191	0.0162	6.17
100	0.0358	0.0315	3.24
50	0.0595	0.0552	1.81
20	0.1120	0.1091	0.92

As a matter of fact, the points indicated by this table lie almost exactly on a line which cuts the x axis at a point the abscissa of which is a little less than forty. The apparent resistance of the galvanometer is, therefore, a trifle less than 40 ohms, while its real resistance with this wire is less than 20 ohms.

The Motion of a Suspended System which carries a relatively Large Damping Vane under Righting Couples of Different Strengths.

In order to study the effects of different restoring moments upon a swinging system furnished with a given damping vane, I used the apparatus represented in Figure 3 (Plate 2). G is a uniformly wound solenoid the horizontal axis of which lies in the meridian at a place where H is known. From a fine fibre in a narrow chimney inserted in the top of the solenoid at the centre hangs a small bar magnet (Q) fastened to a stiff mica vane in the manner shown at N. The axis of the magnet is coincident with the axis of the solenoid. A small mirror on the vertical wire which carries the vane and magnet lies in a vertical plane which makes an angle of 45° with the vertical plane through the axis

of the solenoid, and, receiving the light from a small round hole in a brass plate in the slide holder of a distant Schuckert projecting lantern, throws it upon a sheet of bromide paper wound upon the drum D, where a small very sharp image of the hole is formed. The drum may be turned uniformly at very various speeds, either by clockwork or by an alternating motor actuated by a 60 cycle, 110 volt street circuit. The magnetic field about the suspended magnet can be given any desired value within wide limits by sending through G a suitable steady current from a battery of large storage cells. A current from another similar battery sent through the coil K serves to deflect the magnet out of the meridian against the given restoring field. When the current in K is suddenly interrupted, the suspended system oscillates with continually decreasing amplitude about the horizontal meridian and makes a record of its motion upon the photographic paper. In order that the seam in the paper on the drum may not come at an undesirable place in the record, the break in K's circuit is made automatically by the drum when it reaches a given position, but the system of relays by which this is accomplished is not indicated in the figure.

Experience gained with this apparatus shows that if the original deflection caused by a steady current in K is not more than 5° or 6°, and if the intensity of the magnetic field about the magnet is not too great, the record obtained after K's circuit has been suddenly broken is such that it is possible to draw a curve of the family $y = A \cdot e^{at}$ which shall, within the errors of observation, pass through all the crests of the diagram except the first two or three. We may assume that the motion in a case like this could be mathematically explained on the assumption that a body of fixed moment of inertia (I),— quite different, however, from the moment of inertia of the actual suspended system swinging in vacuo, — is oscillating under the action of the restoring moment due to the magnetic field and a retarding moment equal at every instant to the product of a damping coefficient ($2\,a$) and the angular velocity of the system. If the intensity of the field about the magnet be somewhat changed, I will have nearly its old value, but the damping coefficient, though constant for a given system swinging with a given period, has a new value when the period is changed. The change of the damping coefficient usually follows the direction of the change indicated by Stokes's theoretical treatment of the resistance encountered by a sphere making harmonic oscillations of small amplitude in a viscous liquid. It is usually rather difficult to determine the apparent moment of inertia of the system (I) with accuracy from observations of the period of the oscillations (for there generally is a fixed period), the value of the damping factor, the intensity of the ex-

ternal magnetic field about the magnet, and the moment of the magnet in that field, but such values of I as my observations give do not seem to change in any such manner as Stokes's formula for the sphere demands. Of course the two cases are mathematically quite different.

If the magnetic field about the magnet is relatively intense, and if the original deflection is as great as 10°, the system swings through its position of equilibrium, when it is released, to an elongation on the other side only a fraction (perhaps a half or a quarter) of the original deflection. From this time on the amplitude decreases slowly and regularly, much as in the case figured in Diagram C.

If a seasoned magnet placed in G be subjected to a magnetic field of several units' strength, the magnetic moment changes, and it is necessary to determine the amount of this change with some care if one needs to know the restoring couple which acts upon the swinging system. I have used for measurements of this kind a simple induction-coefficient apparatus shown diagrammatically in Figure 4 (Plate 2). P and Q are two similar solenoids which may be set anywhere on a horizontal east-west track *vw*. O is a mirror magnetometer the deflections of the needle of which can be determined by the telescope and scale (T, S). A horizontal scale *ab* in the meridian carries a wooden holder which contains a seasoned magnet (M_0) protected from sudden temperature changes, in Gauss's B Position with respect to the magnetometer needle. P and Q are so connected in series with a storage battery, a rheostat, and a standard centiamperemeter that a current can be sent in opposite directions through the solenoids; it is then easy, when a current stronger than any to be used in the subsequent determinations is passing through the circuit, to arrange the positions of P and Q near O on *vw*, so that the current shall not affect the needle. After this adjustment has been made, the magnet to be tested is placed in P somewhere near the middle of the solenoid and so near the needle that the latter is deflected off scale, and the wooden holder containing M_0 is placed on *ab* at such a distance from the needle that the latter is brought back exactly to its undeflected position. If then a current of suitable, small intensity be sent through the solenoid circuit, the change of the moment of the magnet in P from M to M' causes a scale reading z owing to a deflection (δ) of the needle; and if the current has not been too strong, this deflection disappears when the circuit is broken. If the field to which the magnet has been exposed has been fairly large, however, the moment is permanently changed by a small amount, and it is then necessary to follow the same magnetic journey in the testing which is to be taken in the damping experiments.

If the distance of the centre of the auxiliary magnet from the centre of the needle is d_0 centimeters, and if l_0 is the half magnetic length of this magnet, the moment of the couple which it exerted upon the needle before the latter was deflected, and which just balanced the moment due to the magnet to be tested, was $\frac{M_0 d_0}{(d^2_0+l^2_0)^{\frac{3}{2}}}$.

When the magnet under the test is removed, the needle deflection (δ_0) caused by the auxiliary magnet alone is usually too large to be easily measured by aid of the telescope and scale; but if this magnet be removed on its track to such a distance that the deflection δ' can be determined, and if the distance between the centres of the magnet and needle is then d',

$$\frac{\tan \delta_0}{\tan \delta'} = \frac{d_0}{(d^2_0 + l^2_0)^{\frac{3}{2}}} \cdot \frac{(d'^2 + l^2_0)^{\frac{3}{2}}}{d'}, \tag{30}$$

and M′ can be determined in terms of M by means of the equation

$$\frac{M' - M}{M} = \frac{\tan \delta}{\tan \delta_0}. \tag{31}$$

VII. The first magnet (Q′) used with this apparatus was about 4.0 centimeters long and weighed about 7 grams. The whole suspended system had a moment of inertia in vacuo almost exactly equal to 43.0, and the magnetic moment of the seasoned magnet (Q′) when placed with its axis perpendicular to the meridian was about 29.8 units. Its induction coefficient under these circumstances was about 0.0242; its moment in a field of 10.37 gausses was 38.7. Most of the records were made with the drum revolving very slowly at the rate of a turn in 348 seconds: the normal length of a record was 479 millimeters. The periodic time of the swinging system varied from 50.8 seconds to 1.20 seconds in the fields actually used. The torsion coefficient of the fibre was under all circumstances here considered much too small to be appreciable.

Figures 5 (Plate **2**) and 6 (Plate **3**) represent oscillations of the suspended system of which the magnet Q' was a part under fields of about 2 and 12 gausses respectively. In the case shown in Figure 7 (Plate **3**) the magnet was deflected through an angle of perhaps 10° and then suddenly released. The record begins at the point **O**, where a nearly straight line indicates that the magnet was on its way through the position of equilibrium and out on the other side to a point corresponding to a deflection of about 2.5°, after which the amplitude

decreased gradually and regularly. The field here was about 19.3 units. Figures 8, 9 (Plate 4) show the effect of suddenly applying a comparatively strong field (14.3 gausses) when the system is already swinging in a field of about 2 units.

The curious irregularity in the spacing of the record in the last diagram after the strong field was applied came from the fact that the magnet was making oscillations in a vertical plane with an amplitude of about 2′. When the system was at rest, the axis of the magnet and the axis of the solenoid were in the same vertical plane but differed from each other in direction by a small fraction of one degree.

To illustrate the fact that in a weak field where the period of the oscillation is long the amplitude of the motion decreases regularly with a practically constant decrement, and that in somewhat stronger fields the departure from this law is nearly inappreciable, except perhaps at the very beginning of the motion, two or three sets of typical measurements will serve. In very strong fields, when the initial deflections are fairly large, the motion cannot be explained with any good approximation to accuracy on the assumption that the air resistance furnishes a couple proportional to the angular velocity.

TABLE VIII.

PERIODIC TIME, 13.2 SECONDS.

Successive Amplitudes.			
Measured.	Computed.	Measured.	Computed.
857	857	302	283
763	760	269	249
680	673	239	222
605	600	213	195
539	533	189	172
480	480	169	160
427	423	150	148
380	374	134	136
339	325		

When the periodic time was as short as 1.2 seconds, a curve of the family $A \cdot e^{-at}$ which passed through the crests of the figure at the middle of the diagram fell distinctly below the crests at the beginning.

From measurements of photographic records taken with Q' for eight different values of the current in the solenoid, the period (T), the damping coefficient ($2a$), the logarithmic decrement (λ) were determined for every case; the intensity of the magnetic field (H) about the magnet was then found by adding the original strength of the field to

TABLE IX.

PERIODIC TIME, 5.20 SECONDS.

Successive Amplitudes.			
Measured.	Computed.	Measured.	Computed.
795	789	390	388
744	740	365	364
696	693	343	341
655	650	320	320
611	610	302	300
574	571	285	281
536	536	266	264
504	502	250	248
474	471	235	232
445	441	220	217
418	417	205	197

that caused by the measured steady current in the solenoid, and a fairly approximate value of the moment of the magnet was computed from the H thus found and the results of measurements made with the magnet in the induction coefficient apparatus described above. When these quantities were known, it was comparatively easy to determine β from the equation $(\pi^2 + \lambda^2)/T^2 = \beta^2$, and then to get an approximate value of the apparent moment of inertia of the swinging system from the formula $I = MHT^2/(\pi^2 + \lambda^2)$. Some of the results obtained by studying many records of the motion of this suspended system are given in the next table.

TABLE X.

Period.	M H.	Damping Coefficient.	Logarithmic Decrement.
15.90	6.38	0.00914	0.0726
13.8	9.86	0.00927	0.0640
9.9	16.64	0.00985	0.0487
8.05	24.2	0.01029	0.0414
7.63	29.9	0.01067	0.0407
2.85	213	0.01467	0.0209
2.18	359	0.01651	0.0180
1.12	1418	0.01907	0.0115

As has been said above, it is possible to obtain from these data values for the apparent moment of inertia of the oscillating system, but since a slight change in any one of several of the quantities measured might introduce a great change in the quantity computed, the results must

be considered rough. Such a change in the intensity of the earth's field as might come from a passing train of electric cars at two hundred yards distance would appreciably affect the first value given.

The results of this computation are respectively 161, 188, 163, 157, 174, 173, 171, 178. So far as one may judge from these and from similar sets obtained with other systems there is no very strong evidence that I changes materially with T, unless it be for extreme values. The damping coefficient is by no means constant, for its value increases rapidly with the restoring force but not according to any easily recognizable law.

VIII. In the next series of experiments with the apparatus represented in Figure 6 Q' was displaced by another small bar magnet 6.0 centimeters long which, when placed perpendicular to the earth's field at room temperature, had a magnetic moment of 101.2 units. This new magnet (Q'') had a moment 129.5 in a field of 9.07 gausses, and a moment 140.2 in a field of 19.93 gausses, when the field was slowly increased. The same mica vane (x) was used as in the work with Q'.

The results of measurements made upon photographic records made with fields of seven different strengths appear in the next table.

TABLE XI.

Period.	M H.	Damping Coefficient.	Logarithmic Decrement.
14.53	12.5	0.0094	0.0683
6.31	68.0	0.0120	0.0379
4.47	123	0.0136	0.0304
2.97	270	0.0158	0.0235
1.81	669	0.0196	0.0177
1.23	1808	0.0222	0.0137
0.81	4396	0.0283	0.0114

At another time a long series of observations were made with the same system, under somewhat different initial circumstances of field and perhaps of moisture in the atmosphere, with the results given below.

TABLE XII.

Period.	I.	Period.	I.
12.30	287	1.60	285
10.44	295	1.27	295
7.90	285	1.13	297
3.29	280	0.98	293
2.38	280	0.81	292
1.92	294		

Here again the apparent moment of inertia is nearly constant but the damping coefficient increases rapidly as the field about the magnet becomes more intense.

Many kinds of physical measurements concern themselves with the behavior of oscillating systems, and it is often necessary to determine what the apparent moment of inertia of a system is if the motion is in air, and what the exact value of the damping coefficient is at any time. If this is not constant throughout the whole motion, — as it should be if it follows the Gaussian law, which assumes the existence of a fixed logarithmic decrement, — it is necessary to find out how it varies with period and amplitude. If one uses a d'Arsonval galvanometer to measure changes of magnetic flux in a large mass of iron, and for reasons of sensitiveness at some point of a hysteresis diagram needs to introduce extra resistance into the circuit or to remove some which is there already, one cannot compute the effect of the change unless one knows, not the real, but the apparent, resistance of the galvanometer coil, and this depends upon the "constants" of the motion which must be determined with some care; it would not be difficult to show that such deviations from the Gaussian law as one frequently encounters in practice need to be carefully taken into account in accurate work. The fact that the swinging system comes to rest in a comparatively short time suggests that the law may not be exactly followed at any part of the motion.

If, then, a swinging magnet or galvanometer coil is exposed to a relatively strong air damping, we must expect that unless the amplitude is very small there will be an appreciable departure from the Gaussian law. If the system be turned out of the position of equilibrium through a considerable angle and then released, it moves rapidly through this position and out on the other side to a new elongation corresponding to a displacement much smaller than the one from which it started; and this modifies profoundly the theories of some ballistic instruments, but after this the subsequent decrease of the amplitude takes place slowly and regularly, accompanied usually by a slowly decreasing logarithmic decrement. For any small number of swings after the first few, however, the constancy of the logarithmic decrement can often be assumed with sufficient accuracy for ordinary purposes.

The moment of inertia of the swinging system cannot as a rule be computed with any fair approximation from a knowledge of the masses and the geometrical dimensions of the bodies of which the system seems to be made up, for a comparatively large mass of air accompanies the visible system and materially increases the inertia.

The apparent moment of inertia of the system seems usually to remain practically unaltered when the moment of the restoring couple which dominates the swings is changed within wide limits, but under these circumstances the coefficient of damping generally increases rapidly as the restoring moment is increased, and the period decreases. If the restoring moment is due to an external field the periodic time remains fairly constant as the amplitude decreases; but if the moment comes from the torsional rigidity of a stiff wire, the period frequently lengthens somewhat as the amplitude grows small.

In case of a d'Arsonval galvanometer coil hung by different pieces of gimp or wire successively, the damping coefficient is practically the same for large differences of period if the resistance of the coil circuit is unchanged; but if this resistance is changed, the damping coefficient changes in a manner to be quantitatively explained by assuming that the coil has an apparent resistance larger than its real resistance. This apparent resistance may be considered as a constant of the coil as long as the level of the instrument is unchanged. If the righting moment of a swinging coil or magnet exposed to air damping is weak and comes from the torsional rigidity of a piece of fine gimp or fibre, the motion often seems to be anomalous because it depends upon obscure elastic changes.

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Cambridge, Mass.

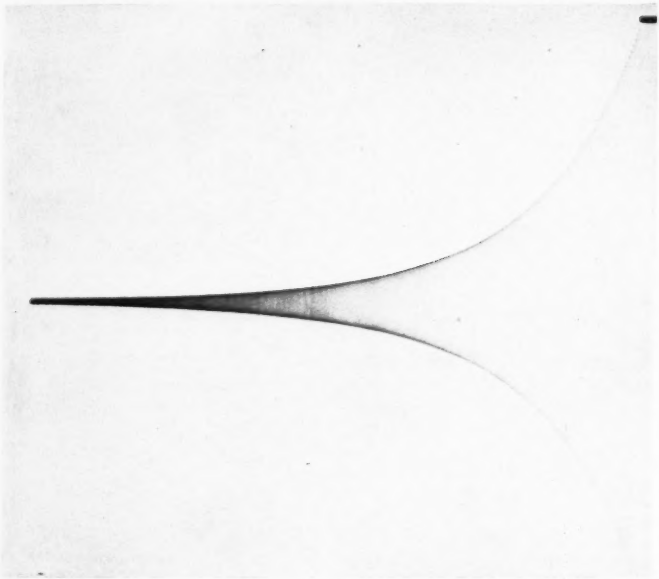


FIGURE 1.

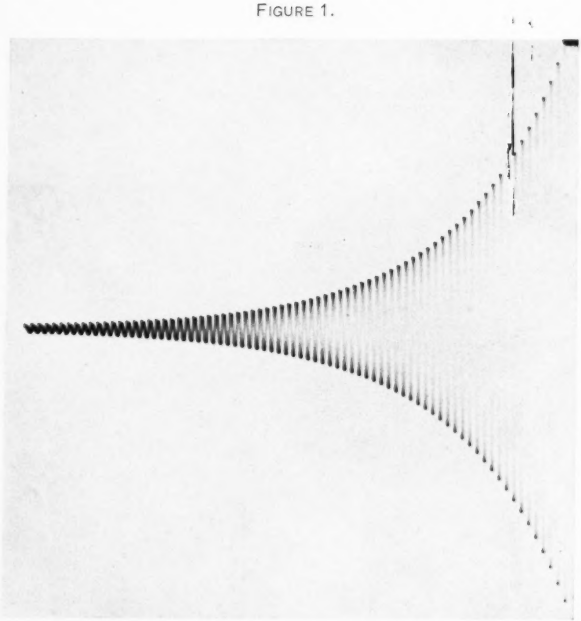


FIGURE 2.

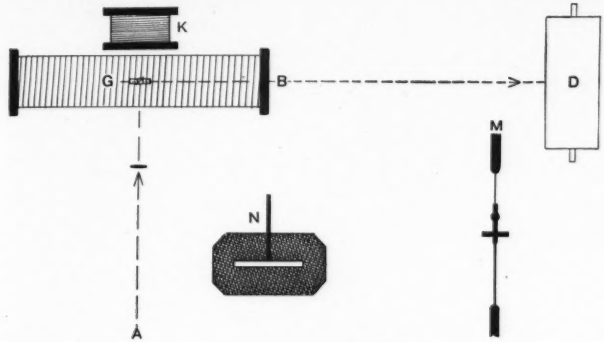


FIGURE 3.

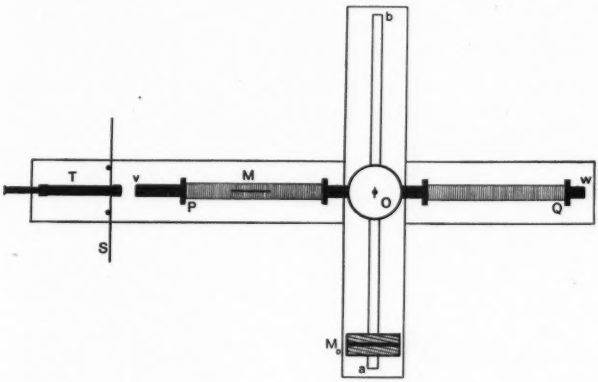


FIGURE . 4

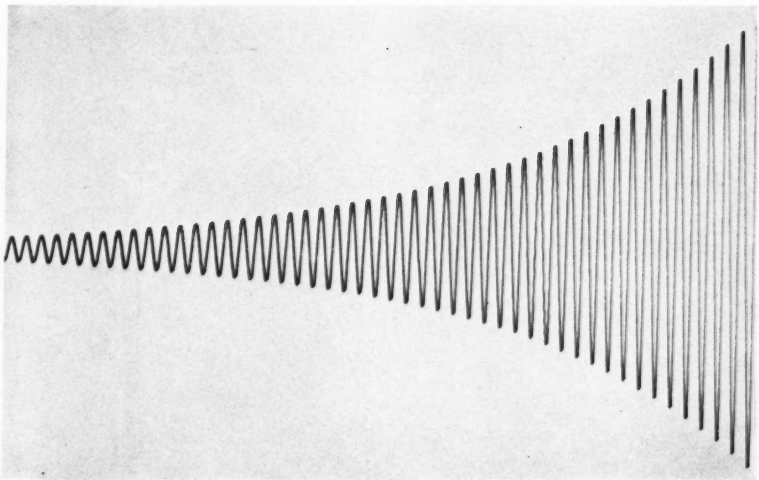
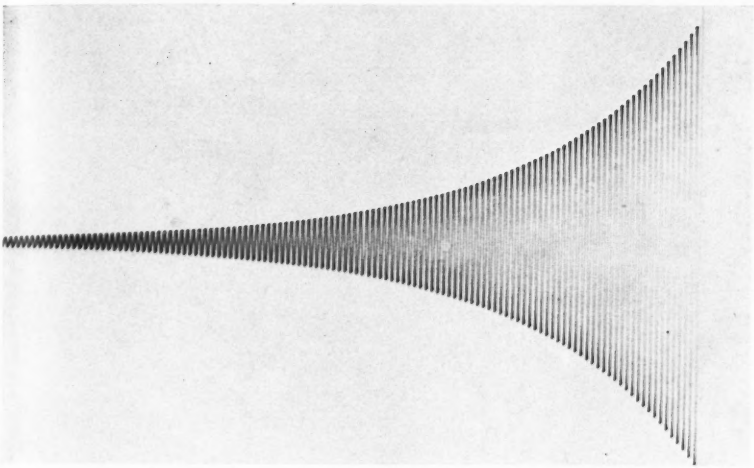


FIGURE 5.



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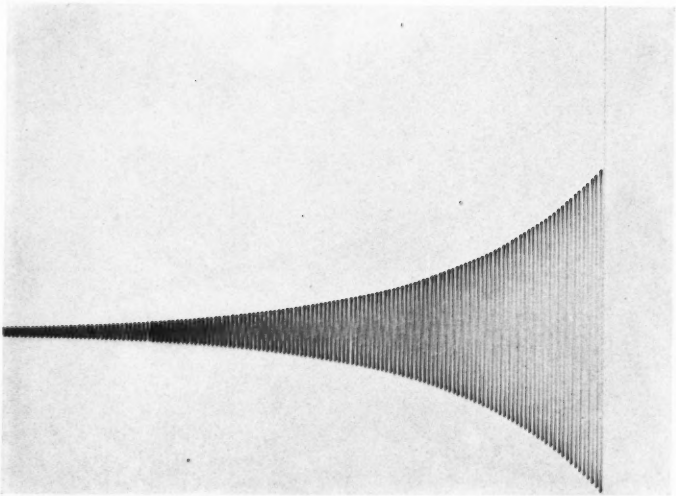

FIGURE 7.

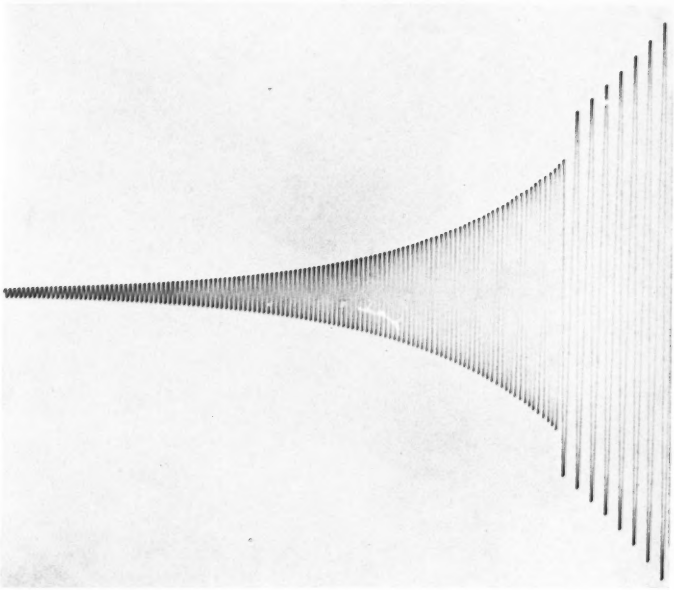

FIGURE 8.

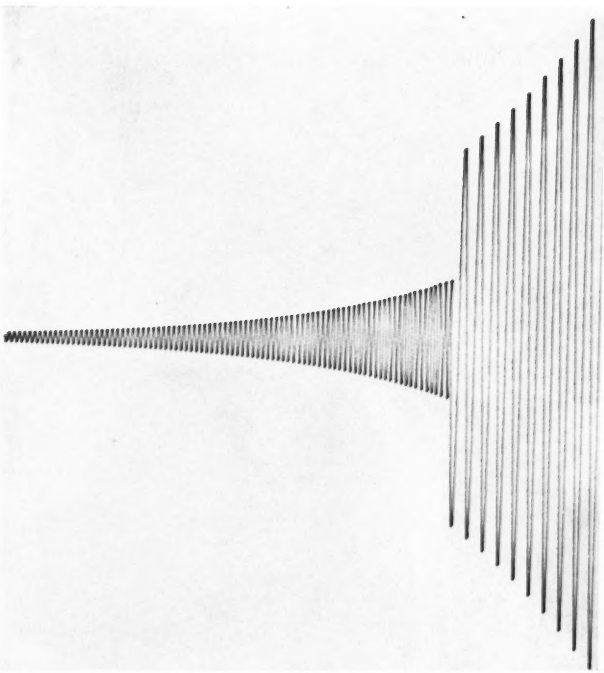

FIGURE 9.